On Harman's "Unreachable Points" Puzzle

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In his blog post "Unreachable points"[1], Radoslav Harman has posed the following puzzle:

> In the 2D plane there is a circular disk D and a point a inside it. For each point b on the boundary of D, and let T_b be the intersection of D and the line passing through the midpoint of the line segment $[a, b]$ and perpendicular to it. What is the set of points of D which do not lie on any of the segments T_b?

In this short note we give a simple proof of a general version of this puzzle.

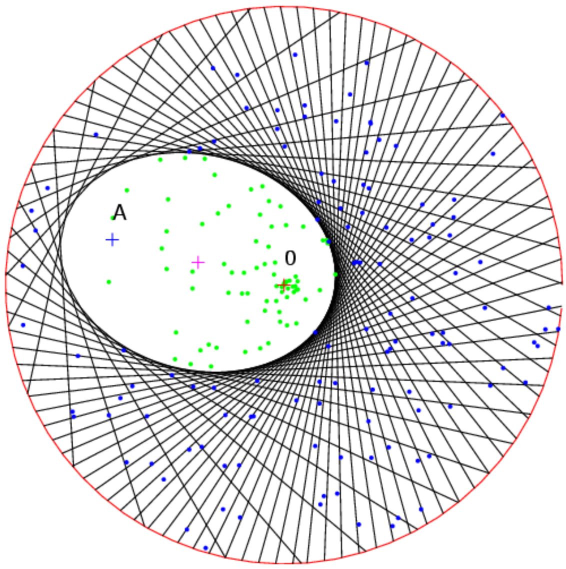

Figure 1: The white shape inside is an ellipsoid with center being the midpoint between a and the center of the circle. The ellipsoid lies inside the disk if a does. The ellipsoid is described by Theorem 5.

Theorem 1. *Let $a \in \mathbf{R}^n$ with $\|a\| \leq 1$. Furthermore, for $b \in \mathbf{R}^n$ let*

$$T_a(b) = \{x \in \mathbf{R}^n \ : \ \langle x - (a+b)/2, a - b\rangle = 0\}.$$

Then

$$S_a \stackrel{def}{=} \bigcap_{\|b\|=1} [T_a(b)]^c = \left\{ x \in \mathbf{R}^n \ : \ \|x\| - \langle a, x\rangle < \frac{1 - \|a\|^2}{2} \right\},$$

where $[T_a(b)]^c = R^n \backslash T_a(b)$, i.e., the complement of $T_a(b)$ in $\mathbf{R}^n$.

[1] http://radoslav-harman.blogspot.com/2010/02/nedosiahnutelne-body.html

Proof. It is easy to see that for $\|b\| = 1$ we have

$$T_a(b) = \left\{ x \in \mathbf{R}^n \;:\; \langle b, x\rangle = \langle a, x\rangle + \frac{1 - \|a\|^2}{2} \right\}. \tag{1}$$

Let us now fix x and ask whether there exists b of unit norm such that $x \in T_a(b)$. It can be shown using a continuity argument together with the Cauchy-Schwarz inequality that the function $b \mapsto \langle b, x\rangle$ maps the unit sphere onto the interval $[-\|x\|, \|x\|]$. This, together with (1) implies such b exists if and only if

$$-\|x\| \le \langle a, x\rangle + \frac{1 - \|a\|^2}{2} \le \|x\|. \tag{2}$$

Since $\langle a, x\rangle + \|x\| \ge -\|a\|\|x\| + \|x\| = \|x\|(1 - \|a\|) \ge 0 \ge (\|a\|^2 - 1)/2$, the left-hand side inequality in (2) is always satisfied. Therefore, x does not lie in $T_a(b)$ for any b of unit norm precisely when the right-hand side inequality in (2) is violated, which proves the theorem. □

Corollary 2. *If $\|a\| < 1$, then all points of S_a have norm strictly less than one. If $\|a\| = 1$, then $S_a = \emptyset$.*

Proof. Assume $\|a\| < 1$, choose $x \in S_a$ and let $x = tx'$, where $t \ge 0$ and $\|x'\| = 1$. Then from Theorem 1 we know that $t\|x'\| - t\langle a, x'\rangle < (1 - \|a\|^2)/2$, and consequently

$$\|x\| = t < \frac{1 - \|a\|^2}{2(1 - \langle a, x'\rangle)} \le \frac{1 - \|a\|^2}{2(1 - \|a\|)} = \frac{1 + \|a\|}{2} < 1.$$

If $\|a\| = 1$, then all $x \in S_a$ must satisfy $\|x\| - \langle a, x\rangle < 0$. Since $\|x\| - \langle a, x\rangle \ge \|x\| - \|a\|\|x\| = 0$, S_a must be empty. □

Corollary 3. *If $\|a\| < 1$, then S_a is a convex set containing 0 and a.*

Proof. It follows from Theorem 1 that S_a is a level set of the convex function $f(x) = \|x\| - \langle a, x\rangle$ and hence it is convex. That $0 \in S_a$ is trivial. To show that $a \in S_a$ it is enough to note that $\|a\| < (1 + \|a\|)/2$ and multiply both sides by $1 - \|a\|$. □

The following is a technical result which we will use in proving that S_a is an ellipsoid.

Lemma 4. *Let $a, x \in \mathbf{R}^n$ satisfy $\|a\| < 1$ and $\langle a, x\rangle + (1 - \|a\|^2)/2 < 0$. Then $\|x\| > -\langle a, x\rangle + (\|a\|^2 - 1)/2$.*

Proof. Let $\alpha = \langle a, x\rangle$. Then $-\alpha = -\langle a, x\rangle \le \|a\|\|x\|$ and hence $\|x\| \ge -\alpha/\|a\|$. It therefore suffices to show that $-\alpha/\|a\| > -\alpha + (\|a\|^2 - 1)/2$, which can be simplified to $\alpha < \frac{1}{2}\|a\|(1 + \|a\|)$. However, we know by assumption that $\alpha < (\|a\|^2 - 1)/2$ and therefore it is enough to prove that $\|a\|^2 - 1 \le \|a\|(1 + \|a\|)$, which is straightforward. □

Theorem 5. *Assume $\|a\| < 1$. Then S_a is a full-dimensional ellipsoid and can be written as*

$$S_a = \{x \in \mathbf{R}^n \;:\; (x - v)^T B (x - v) < r^2\}.$$

Its shape is governed by the positive definite matrix $B = I_n - aa^T$, it has center $v = a/2$, and radius $r = \frac{1}{2}\sqrt{1 - \|a\|^2}$.

Proof. We know from Theorem 1 that $S_a = \{x \in \mathbf{R}^n \ : \ \|x\| < \langle a, x\rangle + (1 - \|a\|^2)/2\}$. Lemma 4 says that we can square both sides of this inequality without having to worry that we have introduced new solutions. Letting $t = \|a\|^2$, we have

$$\begin{aligned} S_a &= \{x \in \mathbf{R}^n \ : \ x^T x < \langle a, x\rangle^2 + (1-t)^2/4 + \langle a, x\rangle(1-t)\} \\ &= \{x \in \mathbf{R}^n \ : \ x^T(I_n - aa^T)x < 2\langle (1-t)a/2, x\rangle + (1-t)^2/4\} \\ &= \{x \in \mathbf{R}^n \ : \ x^T Bx < 2x^T Bv - v^T Bv + v^T Bv + (1-t)^2/4\} \\ &= \{x \in \mathbf{R}^n \ : \ (x-v)^T B(x-v) < v^T Bv + (1-t)^2/4\} \\ &= \{x \in \mathbf{R}^n \ : \ (x-v)^T B(x-v) < r^2\}. \end{aligned}$$

□